Calabi–Yau Conifold Expansions

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Abstract We describe examples of computations of Picard–Fuchs operators for families of Calabi–Yau manifolds based on the expansion of a period near a conifold point. We find examples of operators without a point of maximal unipotent monodromy, thus answering a question posed by J. Rohde.

Key words: Calabi–Yau threefolds, Picard–Fuchs operator, Maximal unipotent monodromy, Conifold point

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1 Introduction

The computation of the instanton numbers n_d for the quintic $X \subset \mathbb{P}^4$ using the period of the quintic mirror Y by P. Candelas, X. de la Ossa and co-workers [10] marked the beginning of intense mathematical interest in the mechanism of mirror symmetry that continues to the present day. On a superficial and purely computational level the calculation runs as follows: one considers the hypergeometric differential operator

$$\mathcal{P} = \theta^4 - 5^5 t(\theta + \tfrac{1}{5})(\theta + \tfrac{2}{5})(\theta + \tfrac{3}{5})(\theta + \tfrac{4}{5})$$

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where $\theta = t\frac{d}{dt}$ denotes the logarithmic derivation. The power series

$$\varphi(t) = \sum \frac{(5n)!}{(n!)^5} t^n$$

is the unique holomorphic solution $\varphi(t) = 1 + \ldots$ to the differential equation

$$\mathcal{P}\varphi = 0.$$

There is a unique second solution ψ that contains a log:

$$\psi(t) = \log(t)\varphi(t) + \rho(t)$$

where $\rho \in t\mathbb{Q}[[t]]$. We now define

$$q := e^{\psi/\varphi} = te^{\rho/\varphi} = t + 770t^2 + \ldots.$$

We can use q as a new coordinate, and as such it can be used to bring the operator $\mathcal{P}$ into the local normal form

$$\mathcal{P} = D^2 \frac{5}{K(q)} D^2$$

where $D = q\frac{d}{dq}$ and $K(q)$ is a power series. When we write this series $K(q)$ in the form of a Lambert series

$$K(q) = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d},$$

one can read off the numbers

$$n_1 = 2875, \quad n_2 = 609250, \quad n_3 = 317206375, \ldots.$$

The data in the calculation are tied to two Calabi–Yau threefolds:

A. The quintic threefold $X \subset \mathbb{P}^4$ ($h^{11} = 1, h^{12} = 101$). The n_d have the interpretation of number of rational degree d curves on X, counted in the Gromov–Witten sense (see [11, 15]).

B. The quintic mirror Y ($h^{11} = 101, h^{12} = 1$). Y is member of a pencil $\mathcal{Y} \longrightarrow \mathbb{P}^1$, and $\mathcal{P}$ is Picard–Fuchs operator of this family. The series φ is the power-series expansion of a special period near the point 0, which is a point of maximal unipotent monodromy, a so-called MUM-point.

As one can see, the whole calculation depends only on the differential operator $\mathcal{P}$ or its holomorphic solution φ and never uses any further geometrical properties of X or Y, except maybe for choice of 5, which is the degree of X.

In [1] this computation was taken as the starting point to investigate so-called *CY3-operators* , which are Fuchsian differential operators $\mathcal{P} \in \mathbb{Q}(t, \theta)$ of order four with the following properties:

1. The operator has the form

$$P = \theta^4 + tP_1(\theta) + \ldots + t^r P_r(\theta)$$

where the P_i are polynomials of degree at most four. This implies in particular that 0 is a MUM-point.
2. The operator $\mathcal{P}$ is *symplectic*. This means the $\mathcal{P}$ leaves invariant a symplectic form in the solution space. The operator than is formally self-adjoint, which can be expressed by a simple condition on the coefficients [1, 7].
3. The holomorphic solution $\varphi(t)$ is in $\mathbb{Z}[[t]]$.
4. Further integrality properties: the expansion of the q-coordinate has integral coefficients, and the instanton numbers are integral (possibly up to a common denominator) [26, 29].

There is an ever-growing list of operators satisfying the first three and probably the last conditions [2]. It starts with the above operator and continues with 13 further hypergeometric cases, which are related to Calabi–Yau threefolds that are complete intersections in weighted projective spaces. Recently, M. Bogner and S. Reiter [7, 8] have classified and constructed the symplectically rigid Calabi–Yau operators, thus providing a solid understanding for the beginning of the list.

Another nice example is operator no. 25 from the list:

$$\mathcal{P} = \theta^4 - 4t(2\theta+1)^2(11\theta^2 + 11\theta + 3) - 16t^2(2\theta+1)^2(2\theta+3)^2.$$

The holomorphic solution of the operator is $\varphi(t) = \sum A_n t^n$ where

$$A_n := \binom{2n}{n}^2 \sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

This operator was obtained in [6] as follows: one considers the Grassmannian $Z := G(2,5)$, a Fano manifold of dimension 6, with $Pic(Z) \approx \mathbb{Z}$, with ample generator h, the class of a hyperplane section in the Plücker embedding. As the canonical class of Z is $-5h$, the complete intersection $X := X(1,2,2)$ by hypersurfaces of degree $1, 2, 2$ is a Calabi–Yau threefold with $h^{11} = 1, h^{12} = 61$. The small quantum cohomology of Z is known, so that one can compute its quantum D-module. The quantum Lefschetz theorem then produces the above operator nr. 25 which thus provides the numbers n_d for X:

$$n_1 = 400, \quad n_2 = 5540, \quad n_3 = 164400, \ \ldots$$

Also, a mirror manifold $Y = Y_t$ was described as (the resolution of the toric closure of) a hypersurface in the torus $(\mathbb{C}^*)^4$ given by a Laurent polynomial.

The question arises which operators in the list are related in a similar way to a mirror pair (X, Y) of Calabi–Yau threefolds with $h^{11}(X) = h^{12}(Y) = 1$. This is certainly not to be expected for all operators, but it suggests the following attractive problem.

Problem. A. Construct examples of Calabi–Yau threefolds X with $h^{11} = 1$ and try to identify the associated quantum differential equation.

B. Construct examples of pencils of Calabi–Yau threefolds $\mathcal{Y} \longrightarrow \mathbb{P}^1$ with $h^{12}(Y_t) = 1$ and try to compute the associated Picard–Fuchs equation.

It has been shown that in many cases one can predict from the operator $\mathcal{P}$ alone topological invariants of X like $(h^3, c_2(X)h, c_3(X))$ [27] and the zeta function of Y_t [23, 30]. In either case we see that the operators of the list provide predictions for the existence of Calabi–Yau threefolds with quite precise properties. Recently, A. Kanazawa [18] has used weighted Pfaffians to construct some Calabi–Yau threefolds X whose existence were predicted in [27]. In this note we report on work in progress to compute the Picard–Fuchs equation for a large number of families of Calabi–Yau threefolds with $h^{12} = 1$.

2 How to Compute Picard–Fuchs Operators

2.1 The Method of Griffiths–Dwork

For a smooth hypersurface $Y \subset \mathbb{P}^n$ defined by a polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ of degree d, one has a useful representation of (the primitive part of) the middle cohomology $H^{n-1}_{prim}(Y)$ using residues of differential forms on the complement $U := \mathbb{P}^n \setminus Y$. One can work with the complex of differential forms with poles along Y and compute modulo exact forms. Although this method was used in the nineteenth century by mathematicians like Picard and Poincaré, it was first developed in full generality by P. Griffiths [16] and B. Dwork [13] in the sixties of the last century.

The *Griffiths' isomorphism* identifies the Hodge space $H^{p,q}_{prim}$ with a graded piece of the Jacobian algebra

$$R := \mathbb{C}[x_0, \dots, x_n]/(\partial_0 F, \partial_1 F, \dots, \partial_n F).$$

More precisely one has

$$\begin{array}{rcl} R_{d(k+1)-(n+1)} & \xrightarrow{\approx} & H^{n-1-k,k}_{prim}(Y) \\ P & \mapsto & Res(\frac{P\Omega}{F^{k+1}}) \end{array}$$

where $\Omega := \iota_E(dx_0 \wedge dx_1 \wedge \dots dx_n)$ and $E = \sum x_i \partial/\partial x_i$ is the Euler vector field. This enables us to find an explicit basis.

If the polynomial F depends on a parameter t, we obtain a pencil $\mathcal{Y} \longrightarrow \mathbb{P}^1$ of hypersurfaces, which can be seen as a smooth hypersurface Y_t over the function field $K := \mathbb{C}(t)$, and the above method provides a basis $\omega_1, \dots, \omega_r$ of differential forms over K. We now can differentiate the differential forms ω_i with respect to t

and express the result in the basis. This step involves a Gröbner-basis calculation. As a result we obtain an $r \times r$ matrix $A(t)$ with entries in K such that

$$\frac{d}{dt}\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_r\end{pmatrix} = A(t)\begin{pmatrix}\omega_1\\ \omega_2\\ \dots\\ \omega_r\end{pmatrix}.$$

The choice of a *cyclic vector* for this differential system then provides a differential operator $\mathcal{P} \in \mathbb{C}\langle t, \theta\rangle$ that annihilates all period integrals $\int_\gamma \omega$. In the situation of Calabi–Yau manifolds there is always a natural vector obtained from the holomorphic differential. For details we refer to the literature, for example [11].

This methods works very well in simple examples and has been used by many authors. It can be generalised to the case of (quasi-)smooth hypersurfaces in weighted projective spaces and more generally complete intersections in toric varieties [4]. Also, it is possible to handle families depending on more than one parameter. A closely related method for tame polynomials in affine space has been implemented by M. Schulze [25] and H. Movasati [21] in Singular. The ultimate generalisation of the method would be an implementation of the *direct image functor in the category of D-modules*, which in principle can be achieved by Gröbner-basis calculations in the Weyl algebra.

The Griffiths–Dwork method however also has some drawbacks:

- In many situations the varieties one is interested in have singularities. For the simplest types of singularities, it is still possible to adapt the method to take the singularities into account, but the procedure becomes increasingly cumbersome for more complicated singularities.
- In many situations the variety under consideration is given by some geometrical construction, and a description with equations seems less appropriate.

In some important situations the following *alternative method* can be used with great success.

2.2 Method of Period Expansion

In order to find Picard–Fuchs operator for a family $\mathcal{Y} \longrightarrow \mathbb{P}^1$, one does the following:

- Find the explicit power-series expansion of a single period

$$\varphi(t) = \int_{\gamma_t} \omega_t = \sum_{n=0}^{\infty} A_n t^n.$$

- Find a differential operator

$$\mathcal{P} = P_0(\theta) + tP_1(\theta) + \ldots + t^r P_r(\theta)$$

that annihilates φ by solving the linear recursion

$$\sum_{i=0}^{r} P_i(n) A_{n-i} = 0$$

on the coefficients. Here the P_i are polynomials in θ of a certain degree d. As $\mathcal{P}$ contains $(d + 1)(r + 1)$ coefficients, we need the expansion of φ only up to sufficiently high order to find it.

This quick-and-dirty method surely is very old and goes back to the time of EULER. And of course, many important issues arise like: *To what order do we need to compute our period?* For this one needs a priori estimates for d and r, which might not be available. Or *Is the operator $\mathcal{P}$ really the Picard–Fuchs operator of the family?* We will not discuss these issues here in detail, as they are not so important in practice: one expands until one finds an operator, and if the monodromy representation is irreducible, the operator obtained is necessarily the Picard–Fuchs operator.

However, it is obvious that the method stands or falls with our ability to find such an explicit period expansion. It appears that the *critical points* of our family provide the clue.

Principle

If one can identify explicitly a vanishing cycle, then its period can be computed "algebraically".

If our family $\mathcal{Y} \longrightarrow \mathbb{P}^1$ is defined over $\mathbb{Q}$, or more generally over a number field, then it is known that such expansions are G-functions and thus have very strong arithmetical properties [3].

Rather than trying to prove here a general statement in this direction, we will illustrate the principle in two simple examples. The appendix contains a general statement that covers the case of a variety acquiring an ordinary double point.

I. Let us look at the LEGENDRE family of elliptic curves given by the equation

$$y^2 = x(t - x)(1 - x).$$

If the parameter t is a small positive real number, the real curve contains a cycle γ_t that runs from 0 to t and back. If we let t go to zero, this loop shrinks to a point and the curve acquires an A_1 singularity. The period of the holomorphic differential $\omega = dx/y$ along this loop is

$$\varphi(t) = \int_{\gamma_t} \omega = 2F(t)$$

where

$$F(t) := \int_0^t \frac{dx}{\sqrt{(x(t-x)(1-x)}}.$$

By the substitution $x \mapsto tx$ we get

$$F(t) = \int_0^1 \frac{1}{\sqrt{(1-xt)}} \frac{dx}{\sqrt{x(1-x)}}.$$

The first square root expands as

$$\frac{1}{\sqrt{(1-xt)}} = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{xt}{4}\right)^n$$

so that

$$F(t) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\int_0^1 \frac{x^n}{\sqrt{x(1-x)}} dx \right) t^n.$$

The appearing integral is well known since the work of Wallis and is a special case of Eulers beta integral.

$$\int_0^1 \frac{x^n}{\sqrt{(x(1-x)}} dx = \pi \binom{2n}{n} \frac{1}{4^n}.$$

So the final result is the beautiful series

$$\begin{aligned} F(t) &= \pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{t}{16}\right)^n \\ &= \pi \left(1 + \left(\frac{1}{2}\right)^2 t + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 t^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 t^3 + \ldots \right). \end{aligned}$$

From this series it is easy to see that the second-order operator with $F(t)$ as solution is

$$4\theta^2 - t(2\theta+1)^2.$$

In fact, the first six coefficients suffice to find the operator.
This should be compared to the Griffiths–Dwork method, which would consist of considering the basis

$$\omega_1 = dx/y, \omega_2 = xdx/y$$

of differential forms on E_t and expressing the derivative

$$\partial_t \omega_1 = -\frac{x(1-x)dx}{(x(t-x)(1-x))^{3/2}}$$

in terms of ω_1, ω_2 modulo exact forms.

II. In mirror symmetry one often encounters families of Calabi–Yau manifolds that arise from a Laurent polynomial

$$f \in \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}].$$

Such a Laurent polynomial f determines a family of hypersurfaces in a torus given by

$$V_t := \{1 - tf(x_1, \ldots, x_n) = 0\} \subset (\mathbb{C}^*)^n.$$

In case the Newton polyhedron $N(f)$ of f is *reflexive*, a crepant resolution of the closure of V_t in the toric manifold determined by $N(f)$ will be a Calabi–Yau manifold Y_t. To compute its Picard–Fuchs operator, the Griffiths–Dwork method is usually cumbersome.

The holomorphic $n-1$-form on Y_t is given on V_t

$$\omega_t := Res_{V_t}\left(\frac{1}{1-tf}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_n}{x_n}\right).$$

There is an $n-1$-cycle γ_t on V_t whose Leray coboundary is homologous to $T := T_\epsilon := \{|x_i| = \epsilon\} \subset (\mathbb{C}^*)^n$. The so-called *principal period* is

$$\varphi(t) = \int_{\gamma_t} \omega_t = \frac{1}{(2\pi i)^n}\int_T \frac{1}{1-tf}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_n}{x_n} = \sum_{n=0}^{\infty}[f^n]_0 t^n$$

where $[g]_0$ denotes the *constant term* of the Laurent series g. For this reason, the series $\varphi(t)$ is sometimes called the *constant term series* of the Laurent polynomial. This method was used in [5] to determine the Picard–Fuchs operator for certain families Y_t and has been popular ever since. A fast implementation for the computation of $[g]_0$ was realised by P. Metelitsyn [19].

3 Double Octics

One of the simplest types of Calabi–Yau threefolds is the so-called *double octic* , which is a double cover Y of $\mathbb{P}^3$ ramified over a surface of degree 8. It can be given by an equation of the form

$$u^2 = f_8(x, y, z, w)$$

and thus can be seen as a hypersurface in weighted projective space $\mathbb{P}(1^4, 4)$. For a general choice of f_8 the variety Y is smooth and has Hodge numbers $h^{11} = 1$, $h^{12} = 149$. A nice subclass of such double octics consists of those for which f_8 is a product of eight planes. In that case Y has singularities at the intersections of the planes. In the generic such situation Y is singular along $8.7/2 = 28$ lines, and by blowing up these lines (in any order), we obtain a smooth Calabi–Yau manifold $\tilde{Y}$

with $h^{11} = 29, h^{12} = 9$. By taking the eight planes in special positions, the double cover Y acquires other singularities, and a myriad of different Calabi–Yau threefolds with various Hodge numbers appear as crepant resolutions $\tilde{Y}$. In [20], 11 configurations leading to rigid Calabi–Yau varieties were identified. Furthermore, C. Meyer listed 63 one-parameter families which thus give 63 special one-parameter families of Calabi– Yau threefolds $\tilde{Y}_t$, and it is for these that we want to compute the associated Picard–Fuchs equation. Due to the singularities of f_8, a Griffiths–Dwork approach is cumbersome, if not impossible. So we resort to the *period expansion method*.

In many of the 63 cases one can identify a *vanishing tetrahedron*: for a special value of the parameter one of the eight planes passes through a triple point of intersection, caused by three other planes. In appropriate coordinates we can write our affine equation as

$$u^2 = xyz(t - x - y - z)P_t(x, y, z)$$

where P_t is the product of the other four planes and we assume $P_0(0, 0, 0) \neq 0$. Analogous to the above calculation with the elliptic curve we now "see" a cycle γ_t, which consists of two copies of the real tetrahedron T_t bounded by the plane $x = 0$, $y = 0$, $z = 0$, $x + y + z = t$. For $t = 0$ the tetrahedron shrinks to a point. So we have

$$\varphi(t) = \int_{\gamma_t} \omega = 2F(t),$$

where

$$F(t) = \int_{T_t} \frac{dxdydz}{\sqrt{xyz(t - x - y - z)P_t(x, y, z)}}.$$

Proposition 1. *The period $\varphi(t)$ expands in a series of the form*

$$\varphi(t) = \pi^2 t(A_0 + A_1 t + A_2 t^2 + \ldots)$$

with $A_i \in \mathbb{Q}$ if $P_t(x, y, z) \in \mathbb{Q}[x, y, z, t]$, $P_0(0, 0, 0) \neq 0$.

Proof. When we replace x, y, z by tx, ty, tz, respectively, we obtain an integral over the standard tetrahedron $T := T_1$:

$$F(t) = t \int_T \frac{dxdydz}{\sqrt{xyz(1 - x - y - z)}} \frac{1}{\sqrt{P_t(tx, ty, tz)}}.$$

We can expand the last square root in a power series

$$\frac{1}{\sqrt{P_t(tx, ty, tz)}} = \sum_{iklm} C_{iklm} x^k y^l z^m t^i$$

and thus find $F(t)$ as a series

$$F(t) = t \sum_{i,k,l,m} \int_T \frac{x^k y^l z^m dxdydz}{\sqrt{xyz(1 - x - y - z)}} C_{iklm} t^i.$$

The integrals appearing in this sum can be evaluated easily in terms of the *generalised beta integral*

$$\int_T x_1^{\alpha_1-1}x_2^{\alpha_2-1}\dots x_n^{\alpha_n-1}(1-x_1-\dots-x_n)^{\alpha_{n+1}-1}dx_1dx_2\dots dx_n$$
$$=\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_{n+1})/\Gamma(\alpha_1+\alpha_2+\dots+\alpha_{n+1}).$$

In particular we get

$$\int_T \frac{x^ky^lz^m dxdydz}{\sqrt{xyz(1-x-y-z)}} = \frac{\Gamma(k+1/2)\Gamma(l+1/2)\Gamma(m+1/2)\Gamma(1/2)}{\Gamma(k+l+m+2)}$$
$$=\pi^2\frac{(2k)!(2l)!(2m)!}{4^{k+l+m}k!l!m!(k+l+m+1)!}\in\pi^2\mathbb{Q}$$

and thus we get an expansion of the form

$$F(t)=\pi^2t(A_0+A_1t+A_2t^2+A_3t^3+\dots)$$

where $A_i\in\mathbb{Q}$ when $P_t(x,y,z)\in\mathbb{Q}[x,y,z,t]$. $\square$

Example 1. Configuration no. 36 of C. Meyer ([20], p. 57) is equivalent to the double octic with equation

$$u^2=xyz(t-x-y-z)(1-x)(1-z)(1-x-y)(1+(t-2)x-y-z).$$

A smooth model has $h^{11}=49, h^{12}=1$. For $t=0$ the resolution is a rigid Calabi–Yau with $h^{11}=50, h^{12}=0$, corresponding to arrangement no. 32. The expansion of the tetrahedral integral around $t=0$ reads

$$F(t)=\pi^2t(1+t+\frac{43}{48}t^2+\frac{19}{24}t^3+\frac{10811}{15360}t^4+\frac{9713}{15360}t^5+\dots).$$

The operator is determined by the first 34 terms of the expansion and reads

$$\begin{aligned}&32\,\theta(\theta-2)(\theta-1)^2-16\,t\theta(\theta-1)\left(9\,\theta^2-13\,\theta+8\right)\\&+8\,t^2\theta\left(33\,\theta^3-32\,\theta^2+38\,\theta-10\right)-t^3(252\,\theta^4+104\,\theta^3+304\,\theta^2+76\,\theta+20)\\&+t^4(132\,\theta^4+224\,\theta^3+292\,\theta^2+160\,\theta+38)\\&-t^5(36\,\theta^4+104\,\theta^3+140\,\theta^2+88\,\theta+21)+4\,t^6(\theta+1)^4\,.\end{aligned}$$

The Riemann symbol of this operator is

$$\left\{\begin{array}{cccc}0&1&2&\infty\\\hline 0&0&0&1\\1&0&0&1\\1&0&2&1\\2&0&2&1\end{array}\right\}.$$

At 0 we have indeed a "conifold point" with its characteristic exponents $0, 1, 1, 2$. At $t = 1$ and $t = \infty$ we find MUM-points. M. Bogner has shown that via a quadratic transformation this operator can be transformed to operator number 10* from the AESZ list, which has Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/256 & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3/2 \end{array}\right\}$$

which is symplectically rigid [8]. So the family of double octics provides a clean B-interpretation for this operator.

Example 2. Configuration no. 70 of Meyer is isomorphic to

$$u^2 = xyz(x+y+z-t)(1-x)(1-z)(x+y+z-1)(x/2+y/2+z/2-1).$$

Again, for general t we obtain a Calabi–Yau threefold with $h^{11} = 49, h^{12} = 1$ and for $t = 0$ we have $h^{11} = 50, h^{12} = 0$, corresponding to the rigid Calabi–Yau of configuration no. 69 of [20]. The tetrahedral integral expands as

$$F(t) = \pi^2 t\left(1 + \frac{13}{16}t + \frac{485}{768}t^2 + \frac{12299}{24576}t^3 + \frac{534433}{1310720}t^4 + \frac{21458473}{62914560}t^5 + \ldots\right)$$

and is annihilated by the operator

$$\begin{aligned} &16\,\theta(\theta-2)(\theta-1)^2 - 2\,t\theta(\theta-1)\left(24\,\theta^2 - 24\,\theta + 13\right) \\ &+t^2\theta^2\left(52\,\theta^2 + 25\right) - 2\,t^3\left(3\,\theta^2 + 3\,\theta + 2\right)(2\,\theta+1)^2 \\ &+t^4(2\,\theta+1)(\theta+1)^2(2\,\theta+3)\,. \end{aligned}$$

The Riemann symbol of this operator is:

$$\left\{\begin{array}{cccc} 0 & 1 & 2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3/2 \end{array}\right\}$$

so we see that it has *no point of maximal unipotent monodromy*!

The first examples of families Calabi–Yau manifolds without MUM-point were described by J. Rohde [22] and studied further by A. Garbagnati and B. van Geemen [14]. It should be pointed out that in those cases the associated Picard–Fuchs operator was of second order, contrary to the above fourth-order operator. M. Bogner has checked that this operator has $Sp_4(\mathbb{C})$ as differential Galois group. It is probably one of the simplest examples of this sort. J. Hofmann has calculated with his package [17] the integral monodromy of the operator. In an appropriate basis it reads

$$T_2 = \begin{pmatrix} 2 & -7 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 7 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 11 & 7 & 2 & 1 \\ -3 & 1 & -1 & 0 \end{pmatrix},$$

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 8 & -16 & 4 & 1 \end{pmatrix}, \quad T_\infty = \begin{pmatrix} 0 & 23 & -3 & -2 \\ 0 & -15 & 2 & 1 \\ 0 & -84 & 11 & 6 \\ 1 & -75 & 11 & 4 \end{pmatrix}$$

with $T_2 T_1 T_0 T_\infty = id$.

As Calabi–Yau operators in the sense of [1] need a to have a MUM, W. Zudilin has suggested to call an operator without such a point of maximal unipotent monodromy an *orphan*.

Example 3. Configuration no. 254 of C. Meyer gives a family of Calabi–Yau threefolds with $h^{11} = 37, h^{12} = 1$:

$$u^2 = xyz(t - x - y - z)P_t(x, y, z)$$

with

$$\begin{aligned} P_t(x, y, z) = {} & (1 - 3z + t - t^2 x + tz - tx - 2y)(1 - z + tx - 2x) \\ & \cdot (1 - tx + z)(1 + t - t^2 x + tz - 5tx + z - 2y - 4x). \end{aligned}$$

For $t = 0$ we obtain the rigid configuration no. 241 with $h^{11} = 40, h^{12} = 0$. The tetrahedral integral expands as

$$F(t) = \pi^2 t \left(1 + \frac{1}{2} t + \frac{37}{24} t^2 + \frac{41}{16} t^3 + \frac{13477}{1920} t^4 + \frac{14597}{768} t^5 + \dots\right).$$

The operator is very complicated and has the following Riemann symbol:

$$\left\{ \begin{array}{ccccccccc} \alpha_1 & \alpha_2 & 0 & \rho_1 & \rho_2 & \rho_3 & -1 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 3/2 \\ 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 3/2 \\ 2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 3/2 \end{array} \right\}$$

where at 0 and $\alpha_{1,2} = -2 \pm \sqrt{5}$ we find conifold points, at the $\rho_{1,2,3}$, roots of the cubic equation $2t^3 - t^2 - 3t + 4 = 0$ we have apparent singularities and at $-1, 1$ we find point of maximal unipotent monodromy, which we also find at ∞, after taking a square root. This operator was not known before.

These three examples illustrate the current *win–win–win* aspect of these calculations. It can happen that the operator is known, in which case we get a nice geometric incarnation of the differential equation. It can happen that the operator does not have a MUM-point, in which case we have found a further example of family of Calabi–Yau threefolds without a MUM-point. From the point of mirror symmetry these cases are of special importance, as the torus for the SYZ fibration, which in the ordinary cases vanishes at the MUM-point, is not in sight. Or it can happen that we find a *new* operator with a MUM-point, thus extending the AESZ table [2].

Many more examples have been computed, in particular also for other types of families, like fibre products of rational elliptic surfaces of the type considered by C. Schoen [24]. The first example of $Sp_4(\mathbb{C})$-operators without MUM-point was found among these [28]. A paper collecting our results on periods of double octics and fibre products is in preparation [12].

4 An Algorithm

Let $\mathcal{Y}$ be a smooth variety of dimension n and $f : \mathcal{Y} \longrightarrow \mathbb{P}^1$ a nonconstant map to $\mathbb{P}^1$ and let $P \in \mathcal{Y}$ be a critical point. In order to analyse the local behaviour of periods of cycles vanishing at P, we replace $\mathcal{Y}$ by an affine part, on which we have a function $f : \mathcal{Y} \longrightarrow \mathbb{A}^1$, with $f(P) = 0$. An n-form

$$\omega \in \Omega^n_{\mathcal{Y},P}$$

gives rise to a family of differential forms on the fibres of f:

$$\omega_t := Res_{Y_t}\left(\frac{\omega}{f-t}\right).$$

The period integrals

$$\int_{\gamma_t} \omega_t$$

over cycles γ_t vanishing at P only depend on the class of ω in the *Brieskorn module* at P, which is defined as

$$\mathcal{H}_P := \Omega^n_{\mathcal{Y},P}/df \wedge d\Omega^{n-2}_{\mathcal{Y},P}.$$

If P is an isolated critical point, it was shown in [9] that the completion $\widehat{\mathcal{H}_P}$ is a (free) $\mathbb{C}[[t]]$-module of rank $\mu(f, P)$, the Milnor number of f at P. In particular, if f has an A_1-singularity at P, we have $\mu(f, P) = 1$, and the image of the class of ω under the isomorphism $\widehat{\mathcal{H}_P} \longrightarrow \mathbb{C}[[t]]$ is, up to a factor, just the expansion of the integral of the vanishing cycle. We will now show how one can calculate this with a simple algorithm.

Proposition 2. *If $f : \mathcal{Y} \longrightarrow \mathbb{A}^1$ and the critical point P of type A_1. If $f : \mathcal{Y} \longrightarrow \mathbb{A}^1$, P and $\omega \in \Omega_{\mathcal{Y},P}$ are defined over $\mathbb{Q}$, then the period integral over the vanishing cycle $\gamma(t)$*

$$\varphi(t) = \int_{\gamma(t)} \omega_t$$

has an expansion of the form

$$\varphi(t) = ct^{n/2-1}(1 + A_1 t + A_2 t^2 + \ldots)$$

where

$$c = d\frac{n}{2}\frac{\Gamma(1/2)^n}{\Gamma(\frac{n}{2}+1)}$$

where $d^2 \in \mathbb{Q}$ and the $A_i \in \mathbb{Q}$ can be computed via a simple algorithm.

Proof. As P and f are defined over $\mathbb{Q}$, we may assume that in appropriate formal coordinates x_i on $\mathcal{Y}$, we have $P = 0$, $f(P) = 0$, and the map is represented by a series

$$f = f_2 + f_3 + f_4 + \ldots$$

where f_2 is a nondegenerate quadratic form and the $f_d \in \mathbb{Q}[x_1, \ldots, x_n]$ are homogeneous polynomials of degree d. After a linear coordinate transformation (which may involve a quadratic field extension), we may and will assume that

$$f_2 = x_1^2 + x_2^2 + \ldots + x_n^2.$$

For $t > 0$ small enough, the part of solution set $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f = t\}$ near 0 looks like a slightly bumped sphere $\gamma(t)$ and is close to standard sphere $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f_2 = t\}$. This is the vanishing cycle we want to integrate $\omega_t = Res(\Omega/(f-t))$ over. Note that

$$\int_0^t \int_{\gamma(t)} \omega_t = \int_{\Gamma(t)} \omega$$

where

$$\Gamma(t) = \cup_{s\in[0,t]}\gamma(s) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f \leq t\}$$

is the *Lefschetz thimble*, which is a slightly bumped ball, that is near to the *standard ball*

$$B(t) := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f_2 \leq t\}.$$

The idea is now to change to coordinates that map f into its quadratic part f_2. An automorphism $\varphi : x_i \mapsto y_i$ of the local ring $R := \mathbb{Q}[[x_1, x_2, \ldots, x_n]]$ is given by n-tuples of series $(y_1, y_2, \ldots, y_n)$ with the property that

$$\left|\frac{\partial y}{\partial x}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \notin \mathcal{M} := (x_1, x_2, \ldots, x_n) \subset R.$$

One has the following *Formal Morse Lemma*: there exist an automorphism φ of R such that

$$\varphi(f) = f_2.$$

Such a φ is obtained by an iteration: if

$$f = f_2 + f_k + f_{k+1} + \dots,$$

then we can find an automorphism φ_k such that

$$\varphi_k(f) = f_2 + \tilde{f}_{k+1} + \dots .$$

To find φ_k it is sufficient to write $f_k = \sum a_i \partial f / \partial x_i$ and set $\varphi_k(x_i) = x_i - a_i$.

Alternatively, we may say that one can find formal coordinates $y_i = \varphi(x_i)$ such that

$$f_2 + f_3 + \dots = y_1^2 + y_2^2 + \dots + y_n^2.$$

By the transformation formula for integrals we get

$$\int_{\Gamma(t)} \omega = \int_{B(t)} \varphi^*(\omega).$$

When we write

$$\omega := A(x) dx_1 dx_2 \dots dx_n,$$

then

$$\varphi^*(\omega) = A(x(y)) \left| \frac{\partial x(y)}{\partial y} \right| dy_1 dy_2 \dots dy_n$$

which can be expanded in a series in the coordinates y_i as

$$\varphi^*(\omega) = \sum_\alpha J_\alpha y^\alpha dy_1 dy_2 \dots dy_n$$

where the $J_\alpha \in \mathbb{Q}$. So we get

$$\int_{\Gamma(t)} \omega = \int_{B(t)} \varphi^*(\omega) = \sum_\alpha J_\alpha \int_{B(t)} y^\alpha dy_1 dy_2 \dots dy_n.$$

The integrals

$$I(\alpha) := \int_{B(t)} y^\alpha dy_1 dy_2 \dots dy_n$$

can be reduced to the generalised beta integral, and one has

Lemma 1. *(i)*

$$I(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$$

when some α_i is odd. (ii)

$$I(2k_1, 2k_2, \dots, 2k_n) = \frac{\Gamma(k_1 + 1/2)\Gamma(k_2 + 1/2) \dots \Gamma(k_n + 1/2)}{\Gamma(k_1 + k_2 + \dots + n/2 + 1)} t^{k_1 + k_2 \dots + k_n + n/2}.$$

As a consequence we have

$$\int_{\Gamma(t)} = \sum J_\alpha I(\alpha) t^{k_1+k_2+\ldots+k_n+n/2}$$
$$= I(0) t^{n/2}(1 + a_1 t + a_2 t^2 + \ldots).$$

The coefficient

$$I(0) = \frac{\Gamma(1/2)^n}{\Gamma(n/2+1)}$$

is the volume of the n-dimensional unit ball. As $I(\alpha)/I(0,0,\ldots,0) \in \mathbb{Q}$, we see that the a_i are also in $\mathbb{Q}$.

So we see that the period integral

$$\varphi(t) = \frac{d}{dt}\int_{\Gamma(t)} \omega$$

has, up to a prefactor, a series expansion with rational coefficients that can be computed algebraically be a very simple although memory-consuming algorithm. Pavel Metelitsyn is currently working on an implementation.

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